Progress in numerical implementation of metaplectic geometrical optics

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Abstract

Radiofrequency waves are widely used for auxiliary heating and current drive in fusion plasmas. The design and optimization of such systems is often performed using ray-tracing codes, which rely on the geometrical-optics (GO) approximation. However, GO is known to fail at wave cutoffs and caustics. To accurately model the wave behavior in these regions, more advanced and computationally expensive "full-wave" simulations are typically used, but this is not strictly necessary. A new, generalized formulation, called metaplectic geometrical optics (MGO), has been proposed that reinstates GO near caustics [1]. The MGO framework yields an integral representation of the wave field, but evaluating the corresponding integral in the general case must be done numerically. We present a survey of numerical integration methods for MGO, including Gaussian quadrature and numerical steepest descent. These methods are benchmarked against analytical solutions in special cases when such solutions are available.

Background

- Modeling electromagnetic (EM) waves is a core aspect of fusion research, especially for plasma heating for tokamaks and stellarators.
- Full-wave EM simulations are computationally expensive, so most simulations assume short wavelength Geometrical Optics (GO).
- GO codes can even model mode conversion [2-4], but fail near caustics, including cutoffs, where the wave number k goes to 0.
- A newly developed method, Metaplectic Geometrical Optics (MGO), has been developed that evades these issues by relying on a sequence of phase space transformations such that the caustics are eliminated in the new variables and GO can be reinstated.

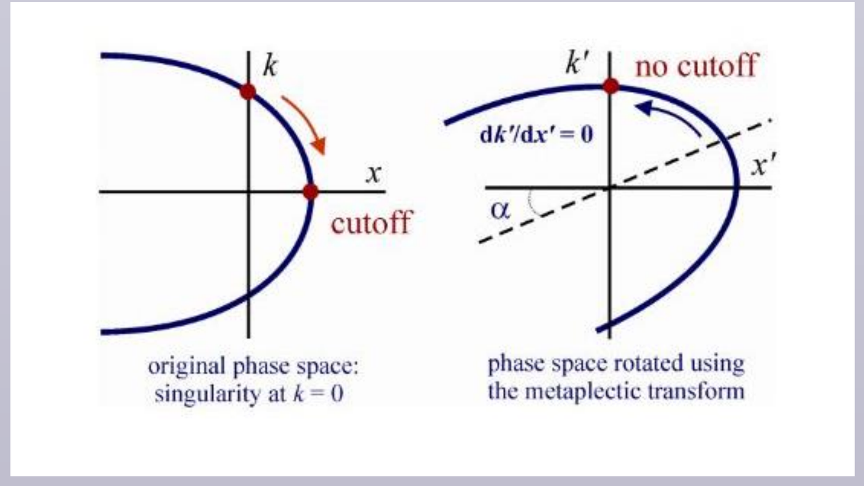
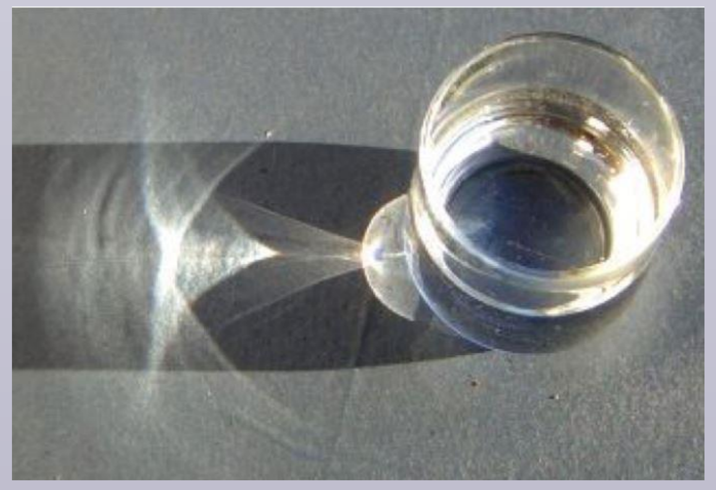


Illustration of phase space rotations used in MGO [5].



The intricate pattern of bright lines are examples of caustics. Image from Wikipedia.

- One relevant physics phenomena to which MGO is applicable is that of a wave incident on a cutoff, which is described by the Airy equation:

$$\frac{d^2}{dx^2}E(x) - xE(x) = 0$$

where $E(x)$ is the electric field.

- MGO requires evaluation of integrals of highly oscillatory functions such as:

$$\Upsilon(p) = \int_{-\infty}^{\infty} d\epsilon \exp\left[ip\epsilon^2 - i\frac{1}{3\vartheta^3}\epsilon^3 - i\frac{p}{\vartheta^6}\epsilon^4\right], \qquad \vartheta \equiv \sqrt{1+4p^2}$$

where $p = 0$ corresponds to a cutoff (reflection point).

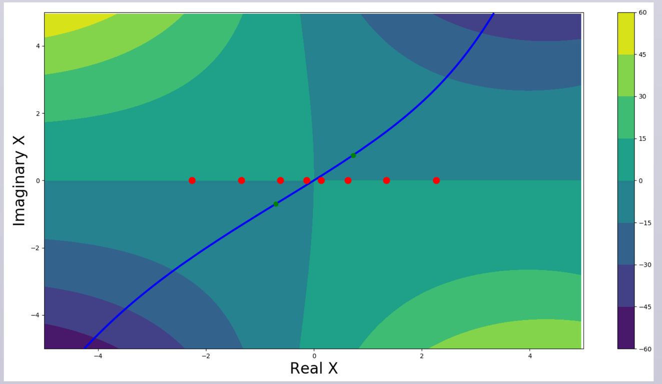
Methodology

- We evaluated all integrals using Gaussian Quadrature (with $n \leq 10$), which is commonly known for its high accuracy. Gaussian Quadrature operates according to the formula:

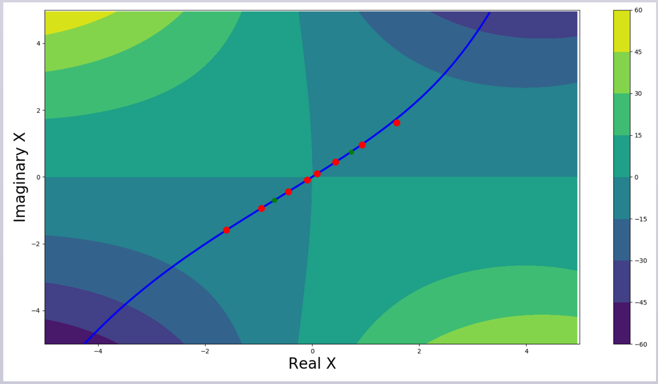
$$\int_a^b \omega(x) f(x) dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

where $\omega(x)$ is the weight function and w_i are the eigenvalues of a matrix distinct to Gaussian Quadrature.

- Initially we used Legendre weights corresponding to $\omega=1$, and Hermite weights, corresponding to $\omega=e^{-x_i^2}$, for both of which the quadrature weights w_i and quadrature points x_i are well−known.
- This worked well for simple test functions, but failed for the highly oscillatory MGO function.
- To fix this, we applied the Method of Steepest Descent.
- This method uses the fact that in the complex plan, the field oscillations are replaced with rapid decrease along certain directions (Steepest Descent Paths).
- The integrals along the complex contours were also taken using Gaussian Quadrature.



Quadrature points (red) when integrating along the real axis. Paths of steepest descent marked by blue lines.



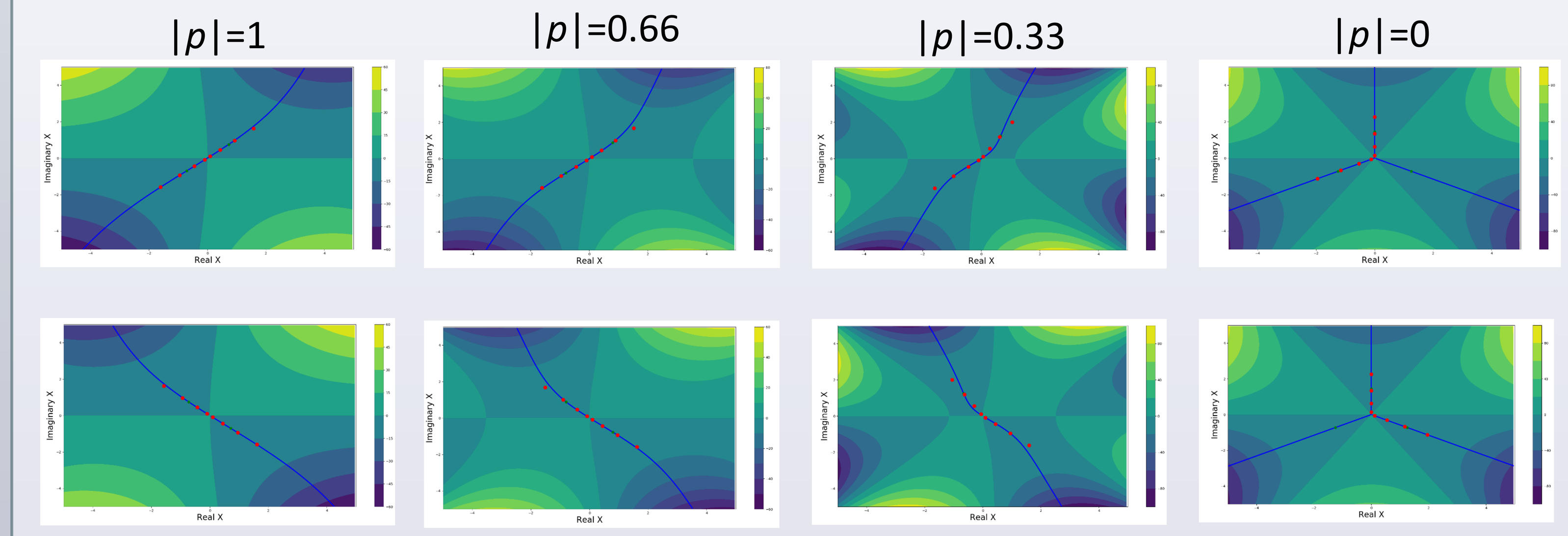
Quadrature points (red) when integrating along straight lines corresponding to the directions of steepest descent (blue) at a saddle point.

- This version was much more accurate for large $|p|$. However, as p decreases, the single-angle rotation method fails due to the incoming and outgoing branches approaching the saddle point at different angles.
- A more advanced quadrature method, using Freud-type weight functions [6,7] was implemented that integrates on $(0,\infty)$ separately on each side from the saddle point. Then, the orientations of the incoming and outgoing paths may differ. The incoming and outgoing branches are then "stitched together".
- The orientation of the integration paths at small p may not coincide with the directions of steepest descent. Therefore, this method selects the corresponding valleys by continuity from the large-p case.
- For each branch, the final quadrature formula is:

$$\int_0^{\infty} e^{-x^2} f(x) dx \approx \sum_{i=1}^{n} w_i e^{i\theta} f(x_i e^{i\theta} + x_0)$$

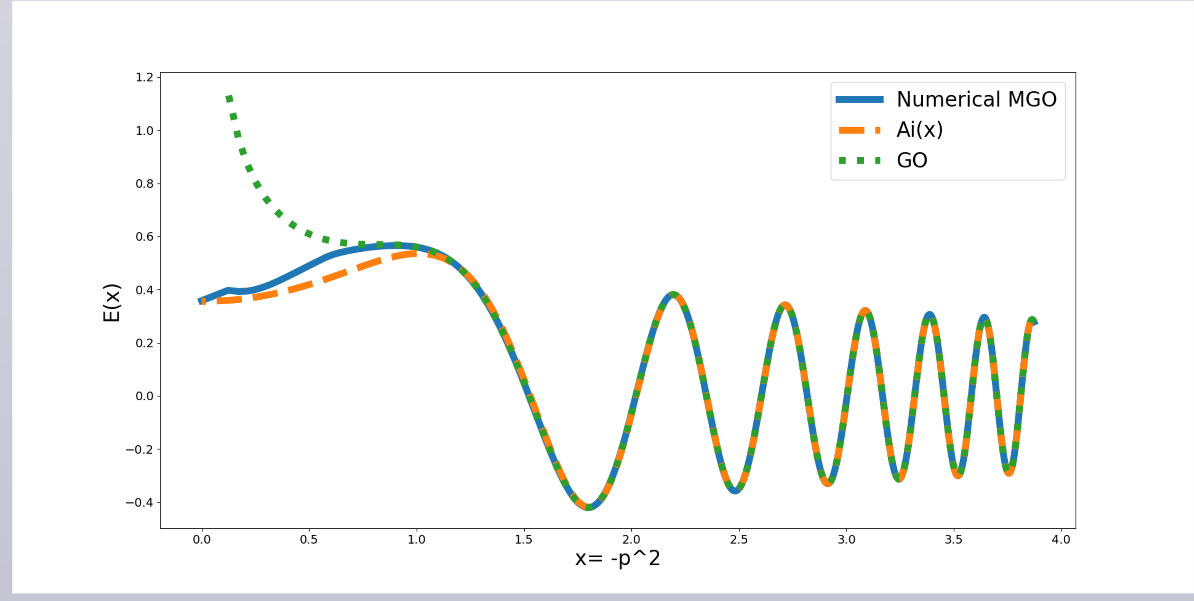
where w_i are the Freud quadrature weights, θ is the angle of rotation, and x_0 is the location of the saddle point.

Results



Evolution of integration contour as $|p|$ decreases from 1 to 0 for positive p (top) and negative p (bottom).

- Our Steepest Descent-Gaussian Quadrature method has been compared with the exact solution for the integral and the GO approximation of this solution.
- MGO solution: $\frac{\Upsilon(p)\exp(-i\frac{2}{3}p^3) + \Upsilon(-p)\exp(i\frac{2}{3}p^3)}{2\pi}$.
- Exact Solution: $\mathrm{Ai}(-p^2)$.
- GO approximation: $\frac{\sin\left(\frac{2}{3}|p|^3 + \frac{\pi}{4}\right)}{\sqrt{\pi|p|}}$.



Electric field vs. the spatial coordinate: numerical results (blue), exact solution Ai(x), and the asymptotic solution.

References

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Acknowledgments

This work was supported in part by the U.S. Department of Energy, Office of Science, Office of Workforce Development for Teachers and Scientists (WDTS) under the Science Undergraduate Laboratory Internship (SULI) program.